

Passive Filters

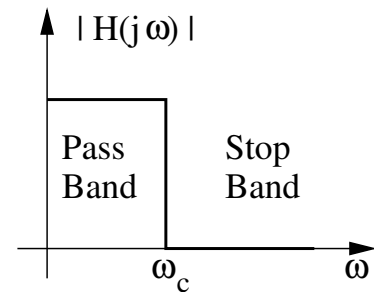
References:

Barbow (pp 265-275), Hayes & Horowitz (pp 32-60), Rizzoni (Chap. 6)

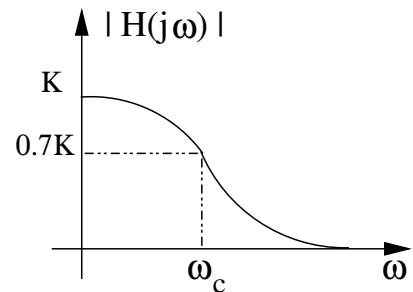
Frequency-selective or filter circuits pass to the output only those input signals that are in a desired range of frequencies (called pass band). The amplitude of signals outside this range of frequencies (called stop band) is reduced (ideally reduced to zero). Typically in these circuits, the input and output currents are kept to a small value and as such, the current transfer function is not an important parameter. The main parameter is the voltage transfer function in the frequency domain, $H_v(j\omega) = V_o/V_i$. Subscript v of H_v is frequently dropped. As $H(j\omega)$ is complex number, it has both a magnitude and a phase, filters in general introduce a phase difference between input and output signals.

Low-Pass Filters

An ideal low-pass filter's transfer function is shown. The frequency between pass and stop bands is called the cut-off frequency (ω_c). All of the signals with frequencies below ω_c are transmitted and all other signals are stopped.

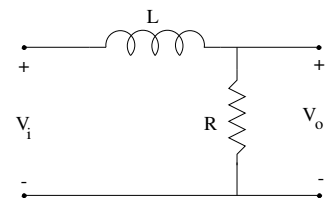


In practical filters, pass and stop bands are not clearly defined, $|H(j\omega)|$ varies continuously from its maximum toward zero. The cut-off frequency is, therefore, defined as the frequency at which $|H(j\omega)|$ is reduced to $1/\sqrt{2} = 0.7$ of its **maximum** value. This corresponds to signal power being reduced by $1/2$ as $P \propto V^2$.



Low-pass RL filters

A series RL circuit as shown acts as a low-pass filter. For no load resistance (output is open circuit, $R_L \rightarrow \infty$), V_o can be found from the voltage divider formula:



$$V_o = \frac{R}{R + j\omega L} V_i \quad \rightarrow \quad H(j\omega) = \frac{V_o}{V_i} = \frac{R}{R + j\omega L} = \frac{1}{1 + j(\omega L/R)}$$

To find the cut-off frequency, we note

$$|H(j\omega)| = \frac{1}{\sqrt{1 + (\omega L/R)^2}}$$

$|H(j\omega)|$ is maximum when denominator is smallest, *i.e.*, $\omega \rightarrow 0$ (alternatively find $d|H(j\omega)|/d\omega$ and set it equal to zero to find $\omega = 0$). In this case,

$$|H(j\omega)|_{max} = 1 \quad \longrightarrow \quad |H(j\omega_c)| = |H(j\omega)|_{\omega=\omega_c} = \frac{1}{\sqrt{2}} |H(j\omega)|_{max} = \frac{1}{\sqrt{2}}$$

$$\frac{1}{\sqrt{1 + (\omega_c L/R)^2}} = \frac{1}{\sqrt{2}} \quad \longrightarrow \quad 1 + \left(\frac{\omega_c L}{R}\right)^2 = 2 \quad \longrightarrow \quad \frac{\omega_c L}{R} = 1$$

Therefore,

$$\omega_c = \frac{R}{L} \quad \text{and} \quad H(j\omega) = \frac{1}{1 + j\omega/\omega_c}$$

Input Impedance: Using the definition of the input impedance, we have:

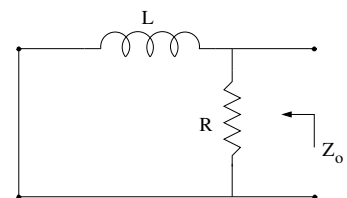
$$Z_i = \frac{V_i}{I_i} = j\omega L + R$$

The value of the input impedance depends on the frequency ω . For good voltage coupling, we need to ensure that the input impedance of this filter is much larger than the output impedance of the previous stage. Thus, the minimum value of Z_i is an important number. Z_i is minimum when the impedance of the inductor is zero ($\omega \rightarrow 0$).

$$Z_i|_{min} = R$$

Output Impedance: The output impedance can be found by “killing” the source and finding the equivalent impedance between output terminals:

$$Z_o = j\omega L \parallel R$$

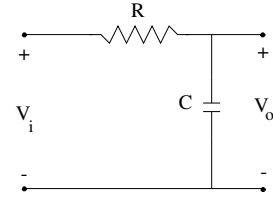


where the source resistance is ignored. Again, the value of the output impedance also depends on the frequency ω . For good voltage coupling, we need to ensure that the output impedance of this filter is much smaller than the input impedance of the next stage, the maximum value of Z_o is an important number. Z_o is maximum when the impedance of the inductor is infinity ($\omega \rightarrow \infty$).

$$Z_o|_{max} = R$$

Low-pass RC filters

A series RC circuit as shown also acts as a low-pass filter. For no load resistance (output is open circuit, $R_L \rightarrow \infty$):



$$V_o = \frac{1/(j\omega C)}{R + 1/(j\omega C)} V_i = \frac{1}{1 + j(\omega RC)} V_i$$

$$H(j\omega) = \frac{1}{1 + j\omega RC}$$

To find ω_c , we follow a procedure similar to RL filters above to find

$$\omega_c = \frac{1}{RC} \quad \text{and} \quad H(j\omega) = \frac{1}{1 + j\omega/\omega_c}$$

similar to the voltage transfer function for low-pass RL filters (only ω_c is different).

Input and Output Impedances: Following the same procedure as for RL filters, we find:

$$Z_i = R + \frac{1}{j\omega C} \quad \text{and} \quad Z_i|_{min} = R$$

$$Z_o = R \parallel \frac{1}{j\omega C} \quad \text{and} \quad Z_o|_{max} = R$$

First-order Low-pass filters

RL and RC filters above are part of the family of first-order filters (they include only one capacitor or inductor). In general, the voltage transfer function of a first-order low-pass filter is in the form:

$$H(j\omega) = \frac{K}{1 + j\omega/\omega_c}$$

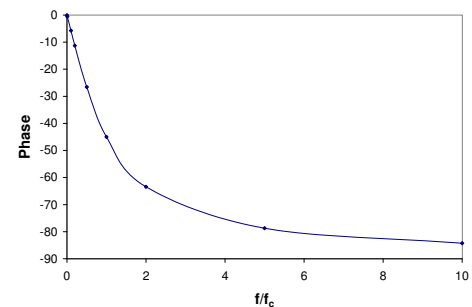
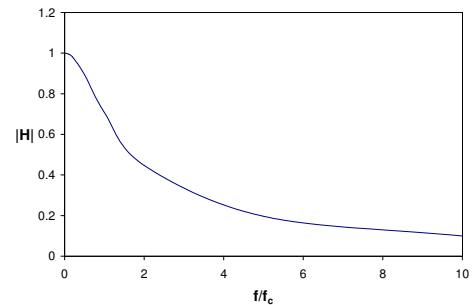
The maximum value of $|H(j\omega)| = K$ is called the filter gain. For RL and RC filters, $K = 1$.

$$|H(j\omega)| = \frac{K}{\sqrt{1 + (\omega/\omega_c)^2}}$$

$$\angle H(j\omega) = -\tan^{-1}\left(\frac{\omega}{\omega_c}\right)$$

For low-pass RL filters: $\omega_c = \frac{R}{L} \quad Z_i|_{min} = R \quad Z_o|_{max} = R$

For low-pass RC filters: $\omega_c = \frac{1}{RC} \quad Z_i|_{min} = R \quad Z_o|_{max} = R$



Bode Plots and Decibel

The ratio of output to input power in a two-port network is usually expressed in Bell:

$$\text{Number of Bels} = \log_{10} \left(\frac{P_o}{P_i} \right) \quad \text{or} \quad \text{Number of Bels} = 2 \log_{10} \left| \frac{V_o}{V_i} \right|$$

because $P \propto V^2$. Bel is a large unit and decibel (dB) is usually used:

$$\text{Number of decibels} = 20 \log_{10} \left| \frac{V_o}{V_i} \right| \quad \text{or} \quad \left| \frac{V_o}{V_i} \right|_{dB} = 20 \log_{10} \left| \frac{V_o}{V_i} \right|$$

There are several reasons why decibel notation is used:

- 1) Historically, the analog systems were developed first for audio equipment. Human ear “hears” the sound in a logarithmic fashion. A sound which appears to be twice as loud actually has 10 times power, *etc.* Decibel translates the output signal to what ear hears.
- 2) If several two-port network are placed in a cascade (output of one is attached to the input of the next), it is easy to show that the overall transfer function, H , is equal to the product of all transfer functions:

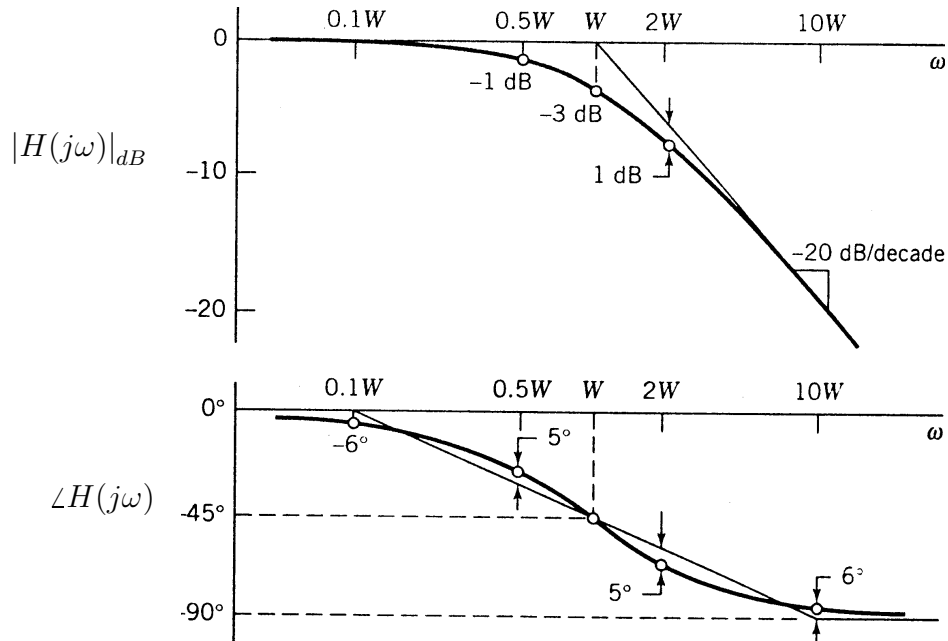
$$\begin{aligned} |H(j\omega)| &= |H_1(j\omega)| \times |H_2(j\omega)| \times \dots \\ 20 \log_{10} |H(j\omega)| &= 20 \log_{10} |H_1(j\omega)| + 20 \log_{10} |H_2(j\omega)| + \dots \\ |H(j\omega)|_{dB} &= |H_1(j\omega)|_{dB} + |H_2(j\omega)|_{dB} + \dots \end{aligned}$$

making it easier to find the overall response of the system.

- 3) Plot of $|H(j\omega)|_{dB}$ versus frequency has special properties that again make analysis simpler as is seen below. For example, using dB definition, we see that, there is 3 dB difference between maximum gain and gain at the cut-off frequency:

$$20 \log |H(j\omega_c)| - 20 \log |H(j\omega)|_{max} = 20 \log \left[\frac{|H(j\omega_c)|}{|H(j\omega)|_{max}} \right] = 20 \log \left(\frac{1}{\sqrt{2}} \right) \approx -3 \text{ dB}$$

Bode plots are plots of $|H(j\omega)|_{dB}$ (magnitude) and $\angle H(j\omega)$ (phase) versus frequency in a semi-log format. Bode plots of first-order low-pass filters ($K = 1$) are shown below (ω_c denotes ω_c).



At high frequencies, $\omega/\omega_c \gg 1$,

$$|H(j\omega)| \approx \frac{1}{\omega/\omega_c} \rightarrow |H(j\omega)|_{dB} = 20 \log \left[\frac{1}{\omega/\omega_c} \right] = 20 \log(\omega_c) - 20 \log(\omega)$$

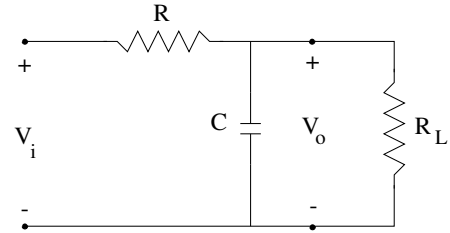
which is a straight line with a slope of -20 dB/decade in the Bode plot. It means that if ω is increased by a factor of 10 (a decade), $|H(j\omega)|_{dB}$ changes by -20 dB.

At low frequencies, $\omega/\omega_c \ll 1$, $|H(j\omega)| \approx 1$ which is also a straight line in the Bode plot. The intersection of these two “asymptotic” values is at $1 = 1/(\omega/\omega_c)$ or $\omega = \omega_c$. Because of this, the cut-off frequency is also called the “corner” frequency.

The behavior of the phase of $H(j\omega)$ can be found by examining $\angle H(j\omega) = -\tan^{-1}(\omega/\omega_c)$. At low frequencies, $\omega/\omega_c \ll 1$, $\angle H(j\omega) \approx 0$ and at high frequencies, $\omega/\omega_c \gg 1$, $\angle H(j\omega) \approx -90^\circ$. At cut-off frequency, $\angle H(j\omega) \approx -45^\circ$.

Terminated RL and RC filters

Terminated two-port networks are referred to those with a finite load resistance. For example, consider this terminated low-pass RC filter:



Voltage Transfer Function: From the circuit,

$$H(j\omega) = \frac{V_o}{V_i} = \frac{1/(j\omega C) \parallel R_L}{R + [1/(j\omega C) \parallel R_L]} = \frac{R'/R}{1 + j(\omega R'C)} \quad \text{with} \quad R' = R \parallel R_L$$

This is similar to the transfer function for unterminated RC filter but with resistance R being replaced by R' . Therefore,

$$\omega_c = \frac{1}{R'C} = \frac{1}{(R \parallel R_L)C} \quad \text{and} \quad H(j\omega) = \frac{R'/R}{1 + j\omega/\omega_c}$$

We see that the impact of the load is to reduce the filter gain ($K = R'/R < 1$) and to shift the cut-off frequency to a higher frequency as $R' = R \parallel R_L < R$.

Input Impedance: $Z_i = R + \frac{1}{j\omega C} \parallel R_L$ $Z_i|_{min} = R$

Output Impedance: $Z_o = R \parallel \frac{1}{j\omega C}$, $Z_o|_{max} = R$

As long as $R_L \gg Z_o$ or $R_L \gg Z_o|_{max} = R$ (our condition for good voltage coupling), $R' \approx R$ and the terminated RC filter will look exactly like an unterminated filter – The filter gain is one, the shift in cut-off frequency disappears, and input and output resistance become the same as before.

Terminated RL low-pass filters

The parameters of the terminated RL filters can be found similarly:

Voltage Transfer Function: $H(j\omega) = \frac{V_o}{V_i} = \frac{1}{1 + j\omega/\omega_c}$, $\omega_c = (R \parallel R_L)/L$.

Input Impedance: $Z_i = j\omega L + R \parallel R_L$, $Z_i|_{min} = R \parallel R_L$

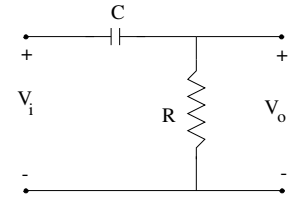
Output Impedance: $Z_o = (j\omega L) \parallel R$, $Z_o|_{max} = R$

Here, the impact of load is to shift the cut-off frequency to a lower value. Filter gain is not affected. Again for $R_L \gg Z_o$ or $R_L \gg Z_o|_{max} = R$ (our condition for good voltage coupling), the shift in cut-off frequency disappears and the filter will look exactly like an unterminated filter.

High-pass RC filters

A series RC circuit as shown acts as a high-pass filter. For no load resistance (output open circuit), we have:

$$H(j\omega) = \frac{V_o}{V_i} = \frac{R}{R + 1/(j\omega C)} = \frac{1}{1 - j(1/\omega RC)}$$



The gain of this filter, $|H(j\omega)|$, is maximum when denominator is smallest, *i.e.*, $\omega \rightarrow \infty$ leading to $|H(j\omega)|_{max} = 1$. Then, the cut-off frequency can be found from

$$|H(j\omega_c)| = \frac{1}{\sqrt{2}} |H(j\omega)|_{max} = \frac{1}{\sqrt{2}}$$

which leads to

$$\omega_c = \frac{1}{RC} \quad H(j\omega) = \frac{1}{1 - j\omega_c/\omega}$$

Input and output impedances of this filter can be found similar to the procedure used for low-pass filters:

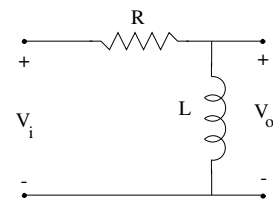
Input Impedance: $Z_i = R + \frac{1}{j\omega C}$ and $Z_i|_{min} = R$

Output Impedance: $Z_o = R \parallel \frac{1}{j\omega C}$ and $Z_o|_{max} = R$

High-pass RL filters

A series RL circuit as shown also acts as a high-pass filter. For no load resistance (output open circuit), we have:

$$\omega_c = \frac{R}{L} \quad H(j\omega) = \frac{1}{1 - j\omega_c/\omega}$$



Input Impedance: $Z_i = R + j\omega L$ and $Z_i|_{min} = R$

Output Impedance: $Z_o = R \parallel j\omega L$ and $Z_o|_{max} = R$

First-order High-pass Filters

In general, the voltage transfer function of a first-order high-pass filter is in the form:

$$H(j\omega) = \frac{K}{1 - j\omega_c/\omega}$$

The maximum value of $|H(j\omega)| = K$ is called the filter gain. For RL and RC high-pass filters, $K = 1$.

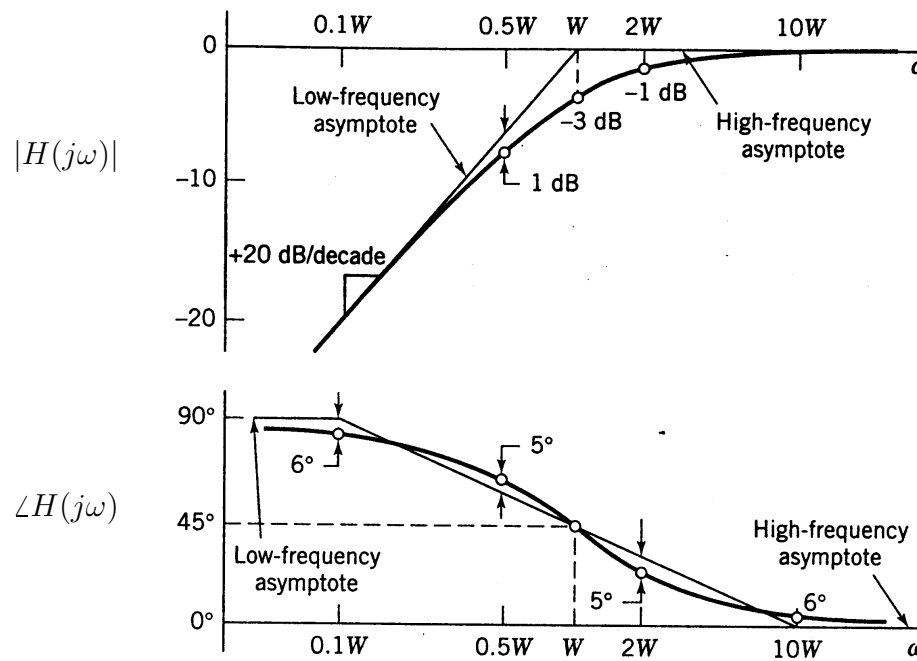
$$|H(j\omega)| = \frac{K}{\sqrt{1 + (\omega_c/\omega)^2}} \quad \angle H(j\omega) = +\tan^{-1}\left(\frac{\omega_c}{\omega}\right)$$

For high-pass RL filters: $\omega_c = \frac{R}{L} \quad Z_{i|_{min}} = R \quad Z_{o|_{max}} = R$

For high-pass RC filters: $\omega_c = \frac{1}{RC} \quad Z_{i|_{min}} = R \quad Z_{o|_{max}} = R$

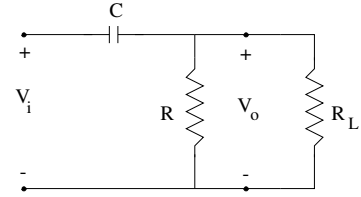
Bode Plots of first-order high-pass filters ($K = 1$) are shown below. The asymptotic behavior of this class of filters is:

At low frequencies, $\omega/\omega_c \ll 1$, $|H(j\omega)| \propto \omega$ (a +20dB/decade line) and $\angle H(j\omega) = 90^\circ$
 At high frequencies, $\omega/\omega_c \gg 1$, $|H(j\omega)| \propto 1$ (a line with a slope of 0) and $\angle H(j\omega) = 0^\circ$



Terminated RC high-pass filters

The parameters of the terminated RC filters can be found similarly:



Voltage Transfer Function: From the circuit,

$$H(j\omega) = \frac{V_o}{V_i} = \frac{R \parallel R_L}{R \parallel R_L + 1/(j\omega C)} = \frac{1}{1 - j(1/\omega R' C)} \quad \text{with} \quad R' = R \parallel R_L$$

This is similar to the transfer function for unterminated RC filter but with resistance R being replaced by R' . Therefore,

$$\omega_c = \frac{1}{R' C} = \frac{1}{(R \parallel R_L) C} \quad \text{and} \quad H(j\omega) = \frac{1}{1 - j\omega_c/\omega}$$

Here, the impact of the load is to shift the cut-off frequency to a higher frequency (as $R' = R \parallel R_L < R$).

$$\text{Input Impedance: } Z_i = \frac{1}{j\omega C} + R \parallel R_L \quad Z_i|_{min} = R \parallel R_L$$

$$\text{Output Impedance: } Z_o = R \parallel \frac{1}{j\omega C} \quad Z_o|_{max} = R$$

As long as $R_L \gg Z_o$ or $R_L \gg Z_o|_{max} = R$ (our condition for good voltage coupling), $R' \approx R$ and the terminated RC filter will look like an unterminated filter – The shift in cut-off frequency disappears and input and output resistance become the same as before.

Terminated RL high-pass filters

The parameters of the terminated RL filters can be found similarly:

$$\text{Voltage Transfer Function: } H(j\omega) = \frac{R'/R}{1 - j\omega_c/\omega} \quad \omega_c = \frac{R'}{L} \quad R' = R \parallel R_L$$

$$\text{Input Impedance: } Z_i = R + (j\omega L) \parallel R_L \quad Z_i|_{min} = R$$

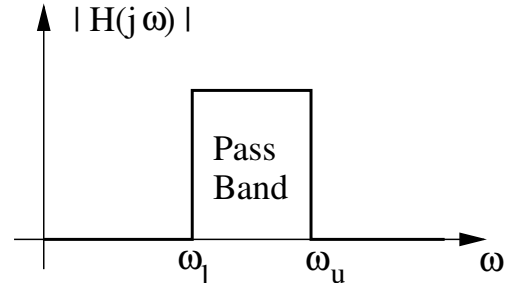
$$\text{Output Impedance: } Z_o = (j\omega L) \parallel R \quad Z_o|_{max} = R$$

We see that the load lowers the gain, $K = R'/R < 1$ and shifts the cut-off frequency to a lower value. As long as $R_L \gg Z_o$ or $R_L \gg Z_o|_{max} = R$ (our condition for good voltage coupling), $R' \approx R$ and the terminated RL filter will look like an unterminated filter.

Band-pass filters

A band pass filter allows signals with a range of frequencies (pass band) to pass through and attenuates signals with frequencies outside this range.

| | |
|--|--------------------------|
| ω_l : | Lower cut-off frequency; |
| ω_u : | Upper cut-off frequency; |
| $\omega_0 \equiv \sqrt{\omega_l \omega_u}$: | Center frequency; |
| $B \equiv \omega_u - \omega_l$: | Band width; |
| $Q \equiv \frac{\omega_0}{B}$: | Quality factor. |



As with practical low- and high-pass filters, upper and lower cut-off frequencies of practical band pass filter are defined as the frequencies at which the magnitude of the voltage transfer function is reduced by $1/\sqrt{2}$ (or -3 dB) from its maximum value.

Second-order band-pass filters:

Second-order band pass filters include two storage elements (two capacitors, two inductors, or one of each). The transfer function for a second-order band-pass filter can be written as

$$H(j\omega) = \frac{K}{1 + jQ \left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega} \right)}$$

$$|H(j\omega)| = \frac{K}{\sqrt{1 + Q^2 \left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega} \right)^2}} \quad \angle H(j\omega) = -\tan^{-1} \left[Q \left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega} \right) \right]$$

The maximum value of $|H(j\omega)| = K$ is called the filter gain. The lower and upper cut-off frequencies can be calculated by noting that $|H(j\omega)|_{max} = K$, setting $|H(j\omega_c)| = K/\sqrt{2}$ and solving for ω_c . This procedure will give two roots: ω_l and ω_u .

$$|H(j\omega_c)| = \frac{1}{\sqrt{2}} |H(j\omega)|_{max} = \frac{K}{\sqrt{2}} = \frac{K}{\sqrt{1 + Q^2 \left(\frac{\omega_c}{\omega_0} - \frac{\omega_0}{\omega_c} \right)^2}}$$

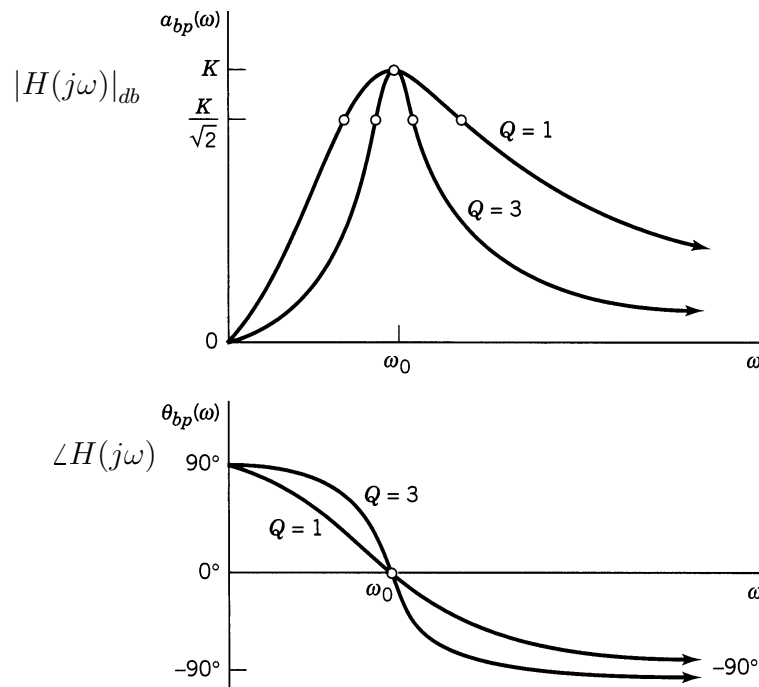
$$Q^2 \left(\frac{\omega_c}{\omega_0} - \frac{\omega_0}{\omega_c} \right)^2 = 1 \quad \rightarrow \quad Q \left(\frac{\omega_c}{\omega_0} - \frac{\omega_0}{\omega_c} \right) = \pm 1$$

$$\omega_c^2 - \omega_0^2 \pm \frac{\omega_c \omega_0}{Q} = 0$$

The above equation is really two quadratic equations (one with + sign in front of fraction and one with a - sign). Solving these equation we will get 4 roots (two roots per equation). Two of these four roots will be negative which are not physical as $\omega_c > 0$. The other two roots are the lower and upper cut-off frequencies (ω_l and ω_u , respectively):

$$\omega_l = \omega_0 \sqrt{1 + \frac{1}{4Q^2}} - \frac{\omega_0}{2Q} \quad \omega_u = \omega_0 \sqrt{1 + \frac{1}{4Q^2}} + \frac{\omega_0}{2Q}$$

Bode plots of a second-order filter is shown below. Note that as Q increases, the bandwidth of the filter become smaller and the $|H(j\omega)|$ becomes more peaked around ω_0 .



Asymptotic behavior:

At low frequencies, $\omega/\omega_0 \ll 1$, $|H(j\omega)| \propto \omega$ (a +20dB/decade line), and $\angle H(j\omega) \rightarrow 90^\circ$

At high frequencies, $\omega/\omega_0 \gg 1$, $|H(j\omega)| \propto 1/\omega$ (a -20dB/decade line), and $\angle H(j\omega) \rightarrow -90^\circ$

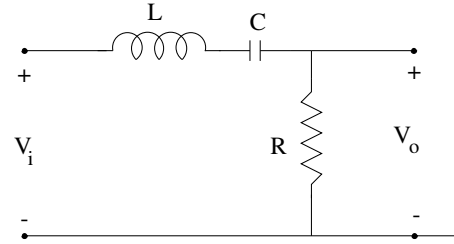
At $\omega = \omega_0$, $H(j\omega) = K$ (purely real) $|H(j\omega)| = K$ (maximum filter gain), and $\angle H(j\omega) = 0^\circ$.

There are two ways to solve second-order filter circuits. 1) One can try to write $H(j\omega)$ in the general form of a second-order filters and find Q and ω_0 . Then, use the formulas above to find the lower and upper cut-off frequencies. 2) Alternatively, one can directly find the upper and lower cut-off frequencies and use $\omega_0 \equiv \sqrt{\omega_l \omega_u}$ to find the center frequency and $B \equiv \omega_u - \omega_l$ to find the bandwidth, and $Q \equiv \omega_0/B$ to find the quality factor. The two examples below show the two methods. Note that one can always find ω_0 and k rapidly as $H(j\omega_0)$ is purely real and $|H(j\omega_0)| = k$

Series RLC Band-pass filters

Using voltage divider formula, we have

$$H(j\omega) = \frac{V_o}{V_i} = \frac{R}{R + j\omega L + 1/(j\omega C)}$$
$$H(j\omega) = \frac{R}{R + j\left(\omega L - \frac{1}{\omega C}\right)}$$



There are two approaches to find filter parameters, K , ω_0 , ω_u , and ω_l .

Method 1: We transform the transfer function in a form similar to general form of the transfer function for second order bandpass filters:

$$H(j\omega) = \frac{K}{1 + jQ\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)}$$

Note that the denominator of the general form is in the form $1 + j\dots$. Therefore, we divide top and bottom of transfer function of series RLC bandpass filters by R :

$$H(j\omega) = \frac{1}{1 + j\left(\frac{\omega L}{R} - \frac{1}{\omega RC}\right)}$$

Comparing the above with the general form of the transfer function, we find $K = 1$. To find Q and ω_0 , we note that the imaginary part of the denominator has two terms, one positive and one negative (or one that scales as ω and the other that scales as $1/\omega$) similar to the general form of transfer function of 2nd-order band-pass filters (which includes $Q\omega/\omega_0$ and $-Q\omega_0/\omega$). Equating these similar terms we get:

$$\frac{Q\omega}{\omega_0} = \frac{\omega L}{R} \quad \rightarrow \quad \frac{Q}{\omega_0} = \frac{L}{R}$$
$$\frac{Q\omega_0}{\omega} = \frac{1}{\omega RC} \quad \rightarrow \quad Q\omega_0 = \frac{1}{RC}$$

We can solve these two equations to find:

$$\omega_0 = \frac{1}{\sqrt{LC}} \quad Q = \frac{\omega_0}{R/L} = \sqrt{\frac{L}{R^2C}}$$

The lower and upper cut-off frequencies can now be found from the formulas on page 24.

Method 2: In this method, we directly calculate the filter parameters similar to the procedure followed for general form of transfer function in page 23. Some simplifications can be made by noting: 1) At $\omega = \omega_0$, $H(j\omega)$ is purely real and 2) $K = H(j\omega = j\omega_0)$.

Starting with the transfer function for the series RLC filter:

$$H(j\omega) = \frac{R}{R + j\left(\omega L - \frac{1}{\omega C}\right)}$$

We note that the transfer function is real if coefficient of j in the denominator is exactly zero (note that this happens for $\omega = \omega_0$), *i.e.*,

$$\omega_0 L - \frac{1}{\omega_0 C} = 0 \quad \longrightarrow \quad \omega_0 = \frac{1}{\sqrt{LC}}$$

Also

$$K = H(j\omega = j\omega_0) = \frac{R}{R} = 1$$

The cut-off frequencies can then be found by setting:

$$|H(j\omega_c)| = \frac{K}{\sqrt{2}} = \frac{1}{\sqrt{2}}$$

$$1 + \left(\frac{\omega_c L}{R} - \frac{1}{\omega_c RC}\right)^2 = 2$$

which can be solved to find ω_u and ω_l similar to page 23.

Input and Output Impedance of band-pass RLC filters

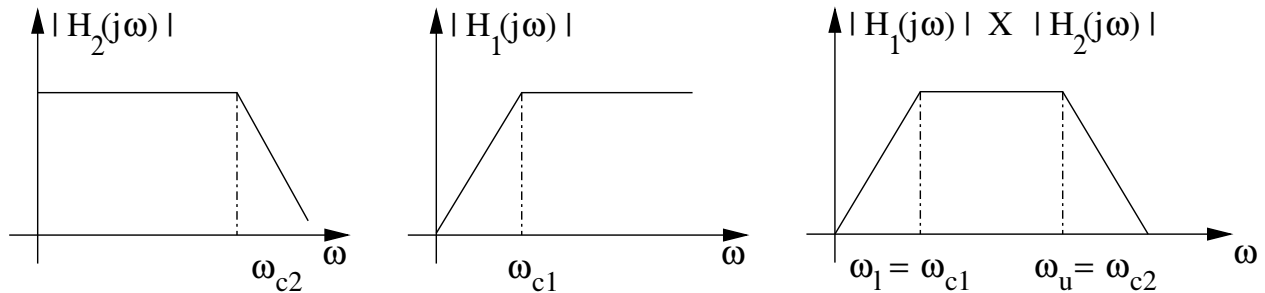
$$Z_i = j\omega L + \frac{1}{j\omega C} + R = j\left(\omega L - \frac{1}{\omega C}\right) + R$$

$$Z_i|_{min} = R \quad \text{occurs at } \omega = \omega_0$$

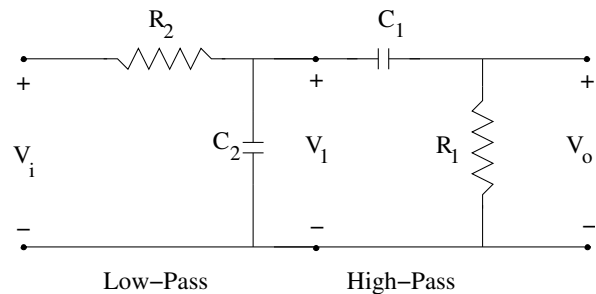
$$Z_o = \left(j\omega L + \frac{1}{j\omega C}\right) \parallel R \quad \longrightarrow \quad Z_o|_{max} = R$$

Wide-Band Band-Pass Filters

Band-pass filters can be constructed by putting a high-pass and a low-pass filter back to back as shown below. The high-pass filter sets the lower cut-off frequency and the low-pass filter sets the upper cut-off frequency of such a band-pass filter.



An example of such a band-pass filter is two RC low-pass and high-pass filters put back to back. These filters are widely used (when appropriate, see below) instead of an RLC filter as inductors are usually bulky and take too much space on a circuit board.



In order to have good voltage coupling in the above circuit, the input impedance of the high-pass filter (actually $Z_i|_{min} = R_1$) should be much larger than the output impedance of the low-pass filter (actually $Z_o|_{max} = R_2$), or we should have $R_1 \gg R_2$. In that case we can use un-terminated transfer functions:

$$H(j\omega) = H_1(j\omega) \times H_2(j\omega) = \frac{1}{1 + j\omega/\omega_{c2}} \times \frac{1}{1 - j\omega_{c1}/\omega}$$

$$\omega_{c1} = 1/(R_1C_1) \quad \omega_{c2} = 1/(R_2C_2)$$

$$H(j\omega) = \frac{1}{(1 + j\omega/\omega_{c2})(1 - j\omega_{c1}/\omega)} = \frac{1}{(1 + \omega_{c1}/\omega_{c2}) + j(\omega/\omega_{c2} - \omega_{c1}/\omega)}$$

Again, we can find the filter parameters by either of two methods above. Transforming the transfer function to a form similar to the general form (left for students) gives:

$$K = \frac{1}{1 + \omega_{c1}/\omega_{c2}} \quad Q = \frac{\sqrt{\omega_{c1}/\omega_{c2}}}{1 + \omega_{c1}/\omega_{c2}} \quad \omega_0 = \sqrt{\omega_{c1}\omega_{c2}}$$

One should note that the Bode plots of previous page are “asymptotic” plots. The real $H(j\omega)$ differs from these asymptotic plots, for example, $|H(j\omega)|$ is 3 dB lower at the cut-off frequency. A comparison of “asymptotic” Bode plots and real ones for first-order high-pass filters are given in page 21. It can be seen that $|H_1(j\omega)|$ achieves its maximum value (1 in this case) only when $\omega/\omega_{c1} < 1/3$. Similarly for the low pass filter, $|H_2(j\omega)|$ achieves its maximum value (1 in this case) only when $\omega/\omega_{c2} > 3$. In the band-pass filter above, if $\omega_{c2} \gg \omega_{c1}$ (i.e., $\omega_{c2} \geq 10\omega_{c1}$), the center frequency of the filter will be at least a factor of three away from both cut-off frequencies and $|H(j\omega)| = |H_1| \times |H_2|$ achieves its maximum value of 1. If ω_{c2} is not $\gg \omega_{c1}$ (i.e., $\omega_{c2} < 10\omega_{c1}$), H_1 and H_2 will not reach their maximum of 1 and the filter $|H(j\omega)|_{\max} = |H_1| \times |H_2|$ will be less than one. This can be seen by examining the equation of K above which is always less than 1 and approaches 1 when $\omega_{c2} \gg \omega_{c1}$.

More importantly, we can never make a “narrow” band filter by putting two first-order high-pass and low-pass filters back to back. When ω_{c2} is not $\gg \omega_{c1}$, $|H(j\omega)|_{\max}$ becomes smaller than 1. Since the cut-off frequencies are located 3 dB below the maximum values, the cut-off frequencies will not be ω_{c1} and ω_{c2} (those frequencies are 3 dB lower than $|H(j\omega)|_{\max} = 1$). The lower cut-off frequency moves to a value lower than ω_{c1} and the upper cut-off frequency moves to a value higher than ω_{c2} . This can be seen by examining the quality factor of this filter at the limit of $\omega_{c2} = \omega_{c1}$

$$Q = \frac{\sqrt{\omega_{c1}/\omega_{c2}}}{1 + \omega_{c1}/\omega_{c2}} = \frac{1}{1 + 1} = 0.5$$

while our asymptotic description of previous page indicated that when $\omega_{c2} = \omega_{c1}$, band-width becomes vanishingly small and Q should become very large.

Because these filters work only when $\omega_{c2} \gg \omega_{c1}$, they are called “wide-band” filters. For these wide-band filters ($\omega_{c1} \ll \omega_{c2}$), we find from above:

$$K = 1 \qquad Q = \sqrt{\omega_{c1}/\omega_{c2}} \qquad \omega_0 = \sqrt{\omega_{c2}\omega_{c1}}$$

$$H(j\omega) = \frac{1}{1 + j(\omega/\omega_{c2} - \omega_{c1}/\omega)}$$

We then substitute for Q and ω_0 in the expressions for cut-off frequencies (page 24) to get:

$$\omega_u = \omega_0 \sqrt{1 + \frac{1}{4Q^2}} + \frac{\omega_0}{2Q} = \frac{\omega_0}{2Q} \left(\sqrt{1 + 4Q^2} + 1 \right)$$

$$\omega_l = \omega_0 \sqrt{1 + \frac{1}{4Q^2}} - \frac{\omega_0}{2Q} = \frac{\omega_0}{2Q} \left(\sqrt{1 + 4Q^2} - 1 \right)$$

Ignoring $4Q^2$ term compared to 1 (because Q is small), we get:

$$\omega_u = \frac{\omega_0}{Q} = \frac{\sqrt{\omega_{c2}\omega_{c1}}}{\sqrt{\omega_{c1}/\omega_{c2}}} = \omega_{c2}$$

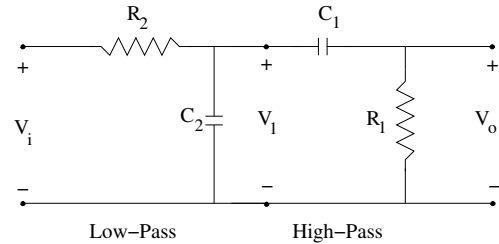
For ω_l , if we ignore $4Q^2$ term compared to 1, we will find $\omega_l = 0$. We should, therefore, expand the square root by Taylor series expansion to get the first order term:

$$\omega_u \approx \frac{\omega_0}{2Q} \left(1 + \frac{1}{2}4Q^2 - 1 \right) = \frac{\omega_0}{2Q} \times 2Q^2 = \omega_0 Q = \omega_{c2}$$

What are Wide-Band and Narrow-Band Filters? Typically, a wide-band filter is defined as a filter with $\omega_{c2} \gg \omega_{c1}$ (or $\omega_{c2} \geq 10\omega_{c1}$). In this case, $Q \leq 0.35$ (prove this!). A narrow-band filter is usually defined as a filter with $B \ll \omega_0$ (or $B \leq 0.1\omega_0$). In this case, $Q \geq 10$.

Example: Design a band-pass filter to pass signals between 160 Hz and 8 kHz. The load for this circuit is 1 M Ω .

As this is wide-band, band-pass filter ($\omega_u/\omega_l = f_u/f_l = 50 \gg 1$), we use two low- and high-pass RC filter stages similar to circuit above. The prototype of the circuit is shown below:



The high-pass filter sets the lower cut-off frequency, and the 1 M Ω load sets the output impedance of this stage. Thus:

$$Z_o|_{max} = R_1 \ll 1 \text{ M}\Omega \quad \rightarrow R_1 \leq 100 \text{ k}\Omega$$

$$\omega_c(\text{High-pass}) = \omega_l = \frac{1}{R_1 C_1} = 2\pi \times 160 \quad \rightarrow R_1 C_1 = 1 \times 10^{-3} \text{ k}\Omega$$

One should choose R_1 as close as possible to 100 k Ω (to make the C_1 small) and $R_1 C_1 = 1 \times 10^{-3}$ using commercial values of resistors and capacitors. A good set here are $R_1 = 100 \text{ k}\Omega$ and $C_1 = 10 \text{ nF}$.

The low-pass filter sets the upper cut-off frequency. The load for this component is the input resistance of the high-pass filter, $Z_i|_{min} = R_1 = 100 \text{ k}\Omega$. Thus:

$$Z_o|_{max} = R_2 \ll 100 \text{ k}\Omega \quad \rightarrow R_2 \leq 10 \text{ k}\Omega$$

$$\omega_c(\text{Low-pass}) = \omega_u = \frac{1}{R_2 C_2} = 2\pi \times 8 \times 10^3 \quad \rightarrow R_2 C_2 = 2 \times 10^{-5}$$

As before, one should choose R_1 as close as possible to 10 k Ω and $R_2 C_2 = 2 \times 10^{-5}$ using commercial values of resistors and capacitors. A good set here are $R_2 = 10 \text{ k}\Omega$ and $C_2 = 2 \text{ nF}$.

In principle, we can switch the position of low-pass and high-pass filter stages in a wide-band, band-pass filter. However, the low-pass filter is usually placed before the high-pass filter because the value of capacitors in such an arrangement will be smaller. (Try redesigning the above circuit with low-pass and high-pass filter stages switched to see that one capacitor become much smaller and one much larger.)